Two projections in a synaptic algebra

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Abstract

We investigate P. Halmos' *two projections theorem*, (or *two subspaces theorem*) in the context of a synaptic algebra (a generalization of the self-adjoint part of a von Neumann algebra).

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1 Introduction

In what follows, A is a synaptic algebra with enveloping algebra $R \supseteq A$, [3, 5, 6, 7, 8, 13] and P is the orthomodular lattice [1, 11] of projections in A. For instance, if $\mathcal{B}(\mathcal{H})$ is the algebra of all bounded linear operators on the Hilbert space $\mathcal{H}$ and $\mathcal{A}$ is the set of all self-adjoint operators in $\mathcal{B}(\mathcal{H})$), then $\mathcal{A}$ is a synaptic algebra with enveloping algebra $\mathcal{B}(\mathcal{H})$. See the literature cited above for numerous additional examples of synaptic algebras.

In this article, we show that Halmos' work [10] on two projections on (or two subspaces of) a Hilbert space can be generalized to the context of the synaptic algebra A.

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A leisurely, lucid, and extended exposition of Halmos' theory of two projections can be found in the paper [2] of A. Böttcher and I.M. Spitkovsky, where the basic theorem [2, Theorem 1.1] is expressed in terms of linear subspaces of a Hilbert space, projections onto these linear subspaces, and operator matrices. Working only with our synaptic algebra A, we have to forgo both Hilbert space and the operator matrix calculus—still we shall formulate generalizations of [10, Theorem 2], often called *Halmos's CS-decomposition theorem.* (See Theorems 5.6, 7.8, and Section 9 below). Also, in Section 8, we give a brief indication of how our version of the CS-decomposition can be used to develop analogues for synaptic algebras of some of the consequences of Halmos' theorem for operator algebras.

2 The orthomodular lattice of projections

In this section we outline some of the notions and facts pertaining to the synaptic algebra A and to the orthomodular lattice $P \subseteq A$ that we shall need in this article. In what follows, we shall use these notions and facts routinely, often without attribution. More details and proofs can be found in [3, 5, 6, 7, 8, 13]. We use the symbol $:=$ to mean 'equals by definition,' as usual we abbreviate 'if and only if' by 'iff,' and the ordered field of real numbers is denoted by $\mathbb{R}$.

If $a, b \in A$, then the product ab, calculated in the enveloping algebra R, may or may not belong to A. However, if $ab = ba$, i.e., if a commutes with b (in symbols aCb), then $ab \in A$. Also, if $ab = 0$, then aCb and $ba = 0$. We define $C(a) := \{b \in A : aCb\}$. If aCc whenever $c \in A$ and cCb, we say that a *double commutes* with b, in symbols $a \in CC(b)$.

There is a unit element $1 \in A$ such that $a = a1 = 1a$ for all $a \in A$. To avoid trivialities, we assume that A is *nondegenerate*, i.e., that $1 \neq 0$.

Let $a, b, c \in A$. Than, although ab need not belong to A, it turns out that $ab + ba \in A$. Likewise, although abc need not belong to A, it can be shown that $abc + cba \in A$.

The synaptic algebra A is a partially ordered real linear space under the partial order relation $\leq$ and we have $0 < 1$ (i.e., $0 \leq 1$ and $0 \neq 1$); moreover, 1 is a (strong) order unit in A. Elements of the "unit interval" $E := \{e \in A : 0 \leq e \leq 1\}$ are called *effects*, and E is a so-called *convex effect algebra* [9].

If $0 \leq a \in A$, then there is a uniquely determined element $r \in A$ such

that $0 \leq r$ and $r^2 = a$; moreover, $r \in CC(a)$ [3, Theorem 2.2]. Naturally, we refer to r as the *square root* of a, in symbols, $a^{1/2} := r$. If $b \in A$, then $0 \leq b^2$, and the *absolute value* of b is defined and denoted by $|b| := (b^2)^{1/2}$. Clearly, $|b| \in CC(b)$ and $|-b| = |b|$. Also, if aCb, then $|a|C|b|$ and $|ab| = |a||b|$.

Elements of the set $P := \{p \in A : p = p^2\}$ are called *projections* and it is understood that P is partially ordered by the restriction of $\leq$. The set P is a subset of the convex set E of effects in A; in fact, P is the extreme boundary of E ([3, Theorem 2.6]). Evidently, $0, 1 \in P$ and $0 \leq p \leq 1$ for all $p \in P$. It turns out that P is a lattice, i.e., for all $p, q \in P$, the *meet* (greatest lower bound) $p \wedge q$ and the *join* (least upper bound) $p \vee q$ of p and q exist in P; moreover, $p \leq q$ iff $pq = qp = p$. Two projections p and q are called *complements* iff $p \wedge q = 0$ and $p \vee q = 1$.

Let $p, q \in P$. The *orthocomplement* of p, defined by $p^\perp := 1 - p$, is again an element of P, and we have the following: $0^\perp = 1$, $1^\perp = 0$, $p^{\perp\perp} = p$, $p \leq q \Rightarrow q^\perp \leq p^\perp$, $p \wedge p^\perp = pp^\perp = 0$, and $p \vee p^\perp = p + p^\perp = 1$. Furthermore, $p \leq q$ iff $q - p \in P$, in which case $q - p = q \wedge p^\perp = qp^\perp = p^\perp q$. Also, we have the *De Morgan laws*: $(p \wedge q)^\perp = p^\perp \vee q^\perp$ and $(p \vee q)^\perp = p^\perp \wedge q^\perp$.

The projections p and q are said to be *orthogonal*, in symbols $p \perp q$, iff $p \leq q^\perp$. The *orthosum* $p \oplus q$ is defined iff $p \perp q$, in which case $p \oplus q := p + q$. It turns out that $p \perp q \Leftrightarrow pCq$ with $pq = qp = 0$; furthermore, $p \perp q \Rightarrow pCq$ with $p \oplus q = p + q = p \vee q \in P$. The lattice P, equipped with the orthocomplementation $p \mapsto p^\perp = 1 - p$, is a so-called *orthomodular lattice* (OML) [1, 11].

2.1 Definition. Following Halmos [10, p. 381], we shall say that two projections $p, q \in P$ are in *generic position* iff

$$p \wedge q = p \wedge q^\perp = p^\perp \wedge q = p^\perp \wedge q^\perp = 0,$$

or equivalently (De Morgan) iff

$$p \vee q = p \vee q^\perp = p^\perp \vee q = p^\perp \vee q^\perp = 1.$$

If $p \in P$ and $e \in E$, then $e \leq p$ iff $e = ep$ iff $e = pe$ [3, Theorem 2.4]. Applying this result to the projection $1 - p$ and the effect $1 - e$, we deduce that $p \leq e$ iff $p = ep$ iff $p = pe$. In particular, if $p, q \in P$, then $p \leq q$ iff $p = pq$ iff $p = qp$.

As is well-known, for projections $p, q \in P$, the question of whether or not pCq can be settled (in various ways) purely in terms of lattice operations in

P. For instance,

$$pCq \Leftrightarrow p = (p \wedge q) \vee (p \wedge q^{\perp}).$$

Using this fact, we obtain the following theorem.

2.2 Theorem. *Let $p, q \in P$ and define $p_r := p \wedge (p^{\perp} \vee q) \wedge (p^{\perp} \vee q^{\perp}) \in P$. Then*:

(i) $p = ((p \wedge q) \vee (p \wedge q^{\perp})) \oplus p_r = (p \wedge q) \vee (p \wedge q^{\perp}) \vee p_r = (p \wedge q) \oplus (p \wedge q^{\perp}) \oplus p_r$.

(ii) $0 \leq p_r \leq p$ *and* $p - p_r = (p \wedge q) \vee (p \wedge q^{\perp})$.

(iii) pCq *iff* $p_r = 0$.

Proof. We have both $p \wedge q \leq p$ and $p \wedge q^{\perp} \leq p$, whence $(p \wedge q) \vee (p \wedge q^{\perp}) \leq p$ and (De Morgan)

$$p - ((p \wedge q) \vee (p \wedge q^{\perp})) = p \wedge ((p \wedge q) \vee (p \wedge q^{\perp}))^{\perp} = p \wedge (p^{\perp} \vee q^{\perp}) \wedge (p^{\perp} \vee q)$$
$$= p \wedge (p^{\perp} \vee q) \wedge (p^{\perp} \vee q^{\perp}) = p_r,$$

whereupon

$$p = ((p \wedge q) \vee (p \wedge q^{\perp})) \oplus p_r = (p \wedge q) \vee (p \wedge q^{\perp}) \vee p_r.$$

Also, $p \wedge q \leq q$ and $p \wedge q^{\perp} \leq q^{\perp}$, so $(p \wedge q) \perp (p \wedge q^{\perp})$ and $(p \wedge q) \vee (p \wedge q^{\perp}) = (p \wedge q) \oplus (p \wedge q^{\perp})$, whence $p = (p \wedge q) \oplus (p \wedge q^{\perp}) \oplus p_r$, completing the proof of (i).

That $0 \leq p_r \leq p$ is clear, $p - p_r = (p \wedge q) \vee (p \wedge q^{\perp})$ follows from (i), and we have (ii). Part (iii) is a consequence of (ii) and the fact that $pCq \Leftrightarrow p = (p \wedge q) \vee (p \wedge q^{\perp})$. $\square$

In view of Theorem 2.2 (iii), we can regard the projection

$$p_r := p \wedge (p^{\perp} \vee q) \wedge (p^{\perp} \vee q^{\perp})$$

as a sort of measure of the extent to which p commutes with q. Indeed, pCq iff $p_r = 0$; also $0 \leq p_r \leq p$ and if $p \neq 0$, then in some sense, the "larger" p_r is, the "greater the lack of commutativity of p and q," culminating in the case in which $p_r = p$. By Theorem 2.2 (ii), $p_r = p$ iff $(p \wedge q) \vee (p \wedge q^{\perp}) = 0$ iff $p \wedge q = p \wedge q^{\perp} = 0$.

An alternative measure of the extent to which p commutes with q, the *Marsden commutator* $[p, q]$, was introduced by E.L. Marsden, Jr. in [12].

2.3 Definition. If $p, q \in P$, then

$$[p, q] := (p \vee q) \wedge (p \vee q^{\perp}) \wedge (p^{\perp} \vee q) \wedge (p^{\perp} \vee q^{\perp}).$$

For the Marsden commutator, we have $0 \leq [p, q] \leq 1$, and as is proved in [12], $pCq \Leftrightarrow [p, q] = 0$. The relationship between $[p, q]$ and the projection p_r in Theorem 2.2 is explicated in Section 3 below. We note that $[p, q]$ is as large as possible, i.e., $[p, q] = 1$, iff p and q are in generic position, a situation which is studied in Sections 6 and 7 below.

We recall some additional basic facts regarding commutativity in P. Let $p, q, r \in P$. If pCq, then $p \wedge q = pq = qp$ and $p \vee q = p + q - pq$. Also, pCq iff $pCq^{\perp}$, and if either $p \leq q$ or $p \perp q$, then pCq. Furthermore, if pCq and pCr, then $pC(q \vee r)$ and $pC(q \wedge r)$. Calculations in the OML P are facilitated by the following theorem [11, Theorem 5, p. 25] which we use routinely in what follows:

2.4 Theorem. *For $p, q, r \in P$, if any two of the relations pCq, pCr, or qCr hold, then $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$ and $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$.*

If $a, b \in A$, then it turns out that $aba \in A$, whence we define the *quadratic mapping* $J_a \colon A \to A$ by $J_a b := aba$ for all $b \in A$. The quadratic mapping J_a is linear and order preserving on A.

A *synaptic automorphism* on A is a mapping $J \colon A \to A$ such that (1) J is a bijection, (2) J is an order automorphism on A, (3) J is a linear automorphism on A, and for all $a, b \in A$, (4) $ab \in A$ iff $JaJb \in A$ and (5) $ab \in A \Rightarrow J(ab) = JaJb$.

An element $u \in A$ is called a *symmetry* [7] iff $u^2 = 1$, and a *partial symmetry* is an element $t \in A$ such that $t^2 \in P$. By the uniqueness theorem for square roots, a projection is the same thing as a partial symmetry p such that $0 \leq p$. Each partial symmetry $t \in A$ has a *canonical extension* to a symmetry $u := t + (t^2)^{\perp}$. If u is a symmetry, then the quadratic mapping J_u, called a *symmetry transformation*, is a synaptic automorphism of A and $J_u^{-1} = J_u$. If u is a symmetry, then so is $-u$, and $J_u = J_{-u}$. If u and v are symmetries, then so are $J_u v = uvu$ and $J_v u = vuv$. By the uniqueness theorem for square roots, if u is a symmetry, then $0 \leq u \Leftrightarrow u = 1$.

Two projections $p, q \in P$ are *exchanged by a symmetry* $u \in A$ iff $J_u p = upu = q$ (whence, automatically, $J_u q = uqu = p$, $J_u p^{\perp} = up^{\perp}u = q^{\perp}$, and $J_u q^{\perp} = uq^{\perp}u = p^{\perp}$). If p and q are exchanged by a symmetry u, then they are also exchanged by the symmetry $-u$. The two projections p and q are

exchanged by a partial symmetry $t \in A$ iff $tpt = q$ and $tqt = p$. If p and q are exchanged by a partial symmetry t and if $u := t + (t^2)^{\perp}$ is the canonical extension of t to a symmetry, then p and q are exchanged by the symmetry u.

Let $a \in A$. Then there is a uniquely determined projection $a^{\mathrm{o}} \in P$, called the *carrier* of a, such that, for all $b \in A$, $ab = 0 \Leftrightarrow a^{\mathrm{o}}b = 0$. It turns out that $a = aa^{\mathrm{o}} = a^{\mathrm{o}}a$, $a^{\mathrm{o}} \in CC(a)$, and a^{o} is the smallest projection $p \in P$ such that $a = ap$ (or, equivalently, $a = pa$). If n is a positive integer, then $(a^n)^{\mathrm{o}} = |a|^{\mathrm{o}} = a^{\mathrm{o}}$. Furthermore, if $b \in A$ and $0 \le a \le b$, then $a^{\mathrm{o}} \le b^{\mathrm{o}}$.

By [3, Definition 4.8, Theorem 4.9 (v), and Theorem 5.6], we have the following result which we shall need in the proof of Lemma 2.12 below and then later in Section 4.

2.5 Lemma. *Let* $p, q \in P$ *and let* $0 \le a \in A$. *Then*:

(i) $(J_p a)^{\mathrm{o}} = (pap)^{\mathrm{o}} = (pa^{\mathrm{o}}p)^{\mathrm{o}} = p \wedge (p^{\perp} \vee a^{\mathrm{o}})$.

(ii) $(pqp)^{\mathrm{o}} = p \wedge (p^{\perp} \vee q)$.

We shall also have use for the next two results which follow from [7, Lemma 4.1] and [8, Theorem 5.5].

2.6 Lemma. *If* $0 \le a_1, a_2, ..., a_n \in A$, *then*

$$(\sum_{i=1}^{n} a_i)^{\mathrm{o}} = \bigvee_{i=1}^{n} (a_i)^{\mathrm{o}}.$$

2.7 Lemma. *If* $a, b, ab \in A$, *then* $(ab)^{\mathrm{o}} = a^{\mathrm{o}}b^{\mathrm{o}} = b^{\mathrm{o}}a^{\mathrm{o}} = a^{\mathrm{o}} \wedge b^{\mathrm{o}}$.

If $a \in A$, there is a partial symmetry $t \in A$, called the *signum* of a, such that $t^2 = a^{\mathrm{o}}$, $t \in CC(a)$, $a = |a|t = t|a|$, and $|a| = ta = at$. If $u := t + (a^{\mathrm{o}})^{\perp}$ is the canonical extension of t to a symmetry, then $u \in CC(a)$, $a = |a|u = u|a|$, and $|a| = ua = au$. The formula $a = |a|u = u|a|$ is referred to as the *polar decomposition* of a.

In Section 4, we shall also need the following theorem [5, Theorem 6.5].

2.8 Theorem. *Let* $p, q \in P$ *and let* $p - q^{\perp} = |p - q^{\perp}|u = u|p - q^{\perp}|$ *be the polar decomposition of* $p - q^{\perp}$, *so that* u *is a symmetry that double commutes with* $p - q^{\perp}$. *Then*: (i) *upqpu=qpq.* (ii) $u(p \wedge (p^{\perp} \vee q))u = q \wedge (p \vee q^{\perp})$.

Consider the synaptic algebra $\mathcal{A}$ of self-adjoint operators on a Hilbert space $\mathcal{H}$. If $B \in \mathcal{A}$, then the carrier B^{o} of B is the projection onto the closure of the range of B. Thus, $B^{\mathrm{o}} = I$, the identity operator on $\mathcal{H}$, iff B is injective and the range of B is dense in $\mathcal{H}$. Also, a symmetry $U \in \mathcal{A}$ is the same thing as a self-adjoint unitary operator on $\mathcal{H}$. Halmos' work in [10] involves unitary operators mapping one linear subspace $\mathcal{M}$ of $\mathcal{H}$ onto another linear subspace $\mathcal{N}$ of $\mathcal{H}$. Assuming that $\mathcal{M}$ and $\mathcal{N}$ are closed, let $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ be the (orthogonal) projections onto $\mathcal{M}$ and $\mathcal{N}$, respectively, and suppose that U is a symmetry in $\mathcal{A}$ that exchanges $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$. Then the restriction $U|\mathcal{M}$ of U to $\mathcal{M}$ is a unitary isomorphism from $\mathcal{M}$ onto $\mathcal{N}$. Thus, in our synaptic algebra A, the situation in which a symmetry u exchanges a projection p with a projection q may be regarded as an analogue of a situation in which two closed subspaces of a Hilbert space are unitarily equivalent via the restriction of a self-adjoint unitary operator.

If n is a positive integer and there are n, but not $n+1$ pairwise orthogonal nonzero projections in P, we say that the synaptic algebra A has *rank n*. On the other hand, if there is an infinite sequence of pairwise orthogonal nonzero projections in P, then we say that A has *infinite rank*. If A is finite dimensional, then it has finite rank, but there are infinite dimensional synaptic algebras of finite rank.

2.9 Remarks. It is not difficult to see that A *is of rank 2 iff every pair of distinct nonorthogonal projections in* $P \setminus \{0, 1\}$ *is in generic position.* In [14, §19], D. Topping introduced the important notion of a *spin factor* and, using the results in [4], it can be shown that a Topping spin factor of dimension greater than 1 is the same thing as synaptic algebra of rank 2. We note that there are infinite-dimensional synaptic algebras of rank 2.

If $r \in P$, then with the partial order and operations inherited from A,

$$rAr := J_r(A) = \{J_r a : a \in A\} = \{rar : a \in A\} = \{b \in A : b = br = rb\}$$

is a synaptic algebra in its own right with rRr as its enveloping algebra and r as its unit element [3, Theorem 4.10]. The orthomodular lattice of projections in rAr is the sublattice $P[0, r] := \{p \in P : p \leq r\}$ of P, and the orthocomplement in $P[0, r]$ of $p \in P[0, r]$ is $p^{\perp_r} := p^{\perp} \wedge r = r - p$. If t is a symmetry in the synaptic algebra rAr, then t is a partial symmetry in A and its canonical extension to a symmetry in A is $u := t + r^{\perp}$. Thus, if $p, q \in P[0, r]$ and if p and q are exchanged by a symmetry t in rAr, then p

and q are exchanged by a symmetry u in A. Let $a \in rAr$. Then, if $0 \leq a$, it follows that $a^{1/2} \in rAr$. Moreover, $|a| \in rAr$, $a^{\rm o} \in rAr$, and $a^{\rm o}$ is the carrier of a as calculated in rAr.

The well-known *Peirce decomposition* of $a \in A$ with respect to $p \in P$, namely

$$a = pap + pap^{\perp} + p^{\perp}ap + p^{\perp}ap^{\perp},$$

is easily proved by direct calculation using the fact that $p^{\perp} = 1 - p$. We note that $pap^{\perp}$ and $p^{\perp}ap$ belong to the enveloping algebra R, but not necessarily to A; however $pap,\ pap^{\perp} + p^{\perp}ap,\ p^{\perp}ap^{\perp} \in A$.

As suggested by the following example, in our work the Peirce decomposition will serve as a substitute for the operator matrix formulas appearing in [2, 10].

2.10 Example. To motivate and illustrate our subsequent work, we consider the case in which $\mathcal{H}$ is a two-dimensional real Hilbert space, A is the rank 2 synaptic algebra of all self-adjoint linear operators on $\mathcal{H}$, $a \in A$, and $p \in P \setminus \{0, 1\}$. Then we can choose an orthonormal basis for $\mathcal{H}$ such that a, p, and $p^{\perp}$, are represented by the matrices

$$a = \begin{bmatrix} \alpha & \gamma \\ \gamma & \beta \end{bmatrix},\ p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ p^{\perp} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

where $\alpha, \beta, \gamma \in \mathbb{R}$. Then, in the Peirce decomposition,

$$pap = \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix},\ pap^{\perp} = \begin{bmatrix} 0 & \gamma \\ 0 & 0 \end{bmatrix},\ p^{\perp}ap = \begin{bmatrix} 0 & 0 \\ \gamma & 0 \end{bmatrix},\ p^{\perp}ap^{\perp} = \begin{bmatrix} 0 & 0 \\ 0 & \beta \end{bmatrix}.$$

2.11 Definition. With the example above in mind, we shall refer to $pap + p^{\perp}ap^{\perp}$ as the *diagonal part* and to $pap^{\perp} + p^{\perp}ap$ as the *off-diagonal part* of $a \in A$ with respect to $p \in P$.

2.12 Lemma. *Let $a \in A$ and $p \in P$. Than*: (i) *If $0 \leq a$, then $a = 0$ iff the diagonal part of a with respect to p is zero.* (ii) *aCp iff the off-diagonal part of a with respect to p is zero.*

Proof. (i) Assume that $pap + p^{\perp}ap^{\perp} = 0$. Then, since $0 \leq a$, we have $pap = 0$ and $p^{\perp}ap^{\perp} = 0$, so $(pap)^{\rm o} = 0$ and $(p^{\perp}ap^{\perp})^{\rm o} = 0$, and it follows from Lemma 2.5 (i) that $p \wedge (p^{\perp} \vee a^{\rm o}) = 0$ and $p^{\perp} \wedge (p \vee a^{\rm o}) = 0$. From $p \wedge (p^{\perp} \vee a^{\rm o}) = 0$, we have $p^{\perp} \vee (p \wedge (a^{\rm o})^{\perp}) = p^{\perp} \oplus (p \wedge (a^{\rm o})^{\perp}) = 1 = p^{\perp} \oplus p$, whence $p \wedge (a^{\rm o})^{\perp} = p$ by cancellation, and we conclude that $p \leq (a^{\rm o})^{\perp}$, i.e., $a^{\rm o} \leq p^{\perp}$. Likewise,

from $p^\perp \wedge (p \vee a^{\mathrm{o}}) = 0$, we deduce that $a^{\mathrm{o}} \leq p$, and it follows that $a^{\mathrm{o}} = 0$, and therefore $a = 0$. The converse is obvious.

(ii) Assume that $pap^\perp + p^\perp ap = 0$. Then $a = pap + p^\perp ap^\perp$, whence $ap = pa = pap$. The converse is obvious. □

3 Two projections—basics

3.1 Standing Assumption. *In what follows, we assume that p and q are arbitrary but fixed projections in the OML P.*

Naturally, the orthocomplements $p^\perp$ and $q^\perp$ will have important roles to play in our subsequent study of p, q, and their mutual interaction. Accordingly, we shall be focusing our attention on the four projections

$$p,\ q,\ p^\perp,\ q^\perp$$

and certain "lattice polynomials" in p, q, $p^\perp$, and $q^\perp$, i.e., projections constructed from these four using lattice meet, join, and orthocomplementation in P.

In the following definition, we extend the notion of the projection r_p in Theorem 2.2 to the projections $r_{p^\perp}$, r_q, and $r_{q^\perp}$. The alternative formulation as a product in each part of the definition is justified by the fact that the projections in each threefold meet commute with one another.

3.2 Definition.

(1) $r_p := p \wedge (p^\perp \vee q) \wedge (p^\perp \vee q^\perp) = p(p^\perp \vee q)(p^\perp \vee q^\perp)$.

(2) $r_{p^\perp} := p^\perp \wedge (p \vee q) \wedge (p \vee q^\perp) = p^\perp(p \vee q)(p \vee q^\perp)$.

(3) $r_q := q \wedge (p \vee q^\perp) \wedge (p^\perp \vee q^\perp) = q(p \vee q^\perp)(p^\perp \vee q^\perp)$.

(4) $r_{q^\perp} := q^\perp \wedge (p \vee q) \wedge (p^\perp \vee q) = q^\perp(p \vee q)(p^\perp \vee q)$.

By Theorem 2.2 and symmetry, we have next two theorems.

3.3 Theorem. *The following conditions are mutually equivalent:*

(i) *At least one of the conditions $r_p = 0$, $r_{p^\perp} = 0$, $r_q = 0$, or $r_{q^\perp} = 0$ holds.*

(ii) $r_p = r_{p^\perp} = r_q = r_{q^\perp} = 0$.

(iii) *At least one of the conditions* pCq, $pCq^\perp$, $p^\perp Cq$, *or* $p^\perp Cq^\perp$ *holds.*

(iv) pCq, $pCq^\perp$, $p^\perp Cq$, *and* $p^\perp Cq^\perp$.

(v) $[p, q] = 0$.

3.4 Theorem.

(i) $p = (p \wedge q) \oplus (p \wedge q^\perp) \oplus r_p$.

(ii) $p^\perp = (p^\perp \wedge q) \oplus (p^\perp \wedge q^\perp) \oplus r_{p^\perp}$.

(iii) $q = (p \wedge q) \oplus (p^\perp \wedge q) \oplus r_q$.

(iv) $q^\perp = (p \wedge q^\perp) \oplus (p^\perp \wedge q^\perp) \oplus r_{q^\perp}$

3.5 Corollary. pCq *iff* $r_p = r_q = 0$ *iff* $r_p C r_q$.

Proof. That $pCq \Rightarrow r_p = r_q = 0$ follows from Theorem 3.3 and obviously $r_p = r_q = 0 \Rightarrow r_p C r_q$. Suppose that $r_p C r_q$. Then in parts (i) and (iii) of Theorem 3.4, every summand in the orthogonal decomposition of p commutes with every summand in the orthogonal decomposition of q, whence pCq. ☐

By parts (ii) and (iii) of the following theorem, the unit element $1 \in A$ is the orthosum, hence also the supremum, (in two different ways) of six pairwise orthogonal projections determined by p and q.

3.6 Theorem.

(i) $r_p \perp r_{p^\perp}$ *and* $r_q \perp r_{q^\perp}$.

(ii) $1 = (p \wedge q) \oplus (p \wedge q^\perp) \oplus (p^\perp \wedge q) \oplus (p^\perp \wedge q^\perp) \oplus r_p \oplus r_{p^\perp} = [p, q]^\perp \oplus r_p \oplus r_{p^\perp}$.

(iii) $1 = (p \wedge q) \oplus (p \wedge q^\perp) \oplus (p^\perp \wedge q) \oplus (p^\perp \wedge q^\perp) \oplus r_q \oplus r_{q^\perp} = [p, q]^\perp \oplus r_q \oplus r_{q^\perp}$.

(iv) $r_p \oplus r_{p^\perp} = r_q \oplus r_{q^\perp} = [p, q]$.

Proof. Part (i) follows from obvious facts that $r_p \le p$, $r_{p^\perp} \le p^\perp$, $r_q \le q$, and $r_{q^\perp} \le q^\perp$.

Part (ii) is a consequence of parts (i) and (ii) of Theorem 3.4, the fact that $1 = p \oplus p^\perp$, and Definition 2.3. Likewise, part (iii) follows from parts (iii) and (iv) of Theorem 3.4 and $1 = q \oplus q^\perp$.

Part (iv) follows from (ii) and (iii). ☐

In the sixfold orthogonal decompositions of the unit 1 in parts (ii) and (iii) of Theorem 3.6, we are inclined to agree with Halmos [10, p. 381] that the first four projections $p \wedge q$, $p \wedge q^{\perp}$, $p^{\perp} \wedge q$, and $p^{\perp} \wedge q^{\perp}$ are "thoroughly uninteresting." What is interesting, is what Halmos refers to as "the rest," namely the projections $r_p \oplus r_{p^{\perp}}$ and $r_q \oplus r_{q^{\perp}}$. Accordingly, in what follows, we pay special attention to the projections r_p, $r_{p^{\perp}}$, r_q, $r_{q^{\perp}}$, and $r_p \oplus r_{p^{\perp}} = r_q \oplus r_{q^{\perp}} = [p, q]$.

3.7 Definition.

(1) $r := r_p \oplus r_{p^{\perp}} = r_q \oplus r_{q^{\perp}} = r_p \vee r_{p^{\perp}} = r_q \vee r_{q^{\perp}} = [p, q]$.

(2) We call the synaptic algebra rAr the *commutator algebra* of p and q.

3.8 Theorem.

(i) *pCq iff $r = 0$.*

(ii) *$pr_p = r_p p = r_p$, $pr_{p^{\perp}} = r_{p^{\perp}} p = 0$, and $pr = rp = p \wedge r = r_p$.*

(iii) *$p^{\perp} r_p = r_p p^{\perp} = 0$, $p^{\perp} r_{p^{\perp}} = r_{p^{\perp}} p^{\perp} = r_{p^{\perp}}$, and $p^{\perp} r = rp^{\perp} = p^{\perp} \wedge r = r_{p^{\perp}}$.*

(iv) *$qr_q = r_q q = r_q$, $qr_{q^{\perp}} = r_{q^{\perp}} q = 0$, and $qr = rq = q \wedge r = r_q$.*

(v) *$q^{\perp} r_q = r_q q^{\perp} = 0$, $q^{\perp} r_{q^{\perp}} = r_{q^{\perp}} q^{\perp} = r_{q^{\perp}}$, and $q^{\perp} r = rq^{\perp} = q^{\perp} \wedge r = r_{q^{\perp}}$.*

(vi) *$p, p^{\perp}, q$, and $q^{\perp}$ commute with $r = r_p \oplus r_{p^{\perp}} = r_q \oplus r_{q^{\perp}} = [p, q]$.*

(vii) *r_p, $r_{p^{\perp}}$, $r_p^{\perp}$, r_q, $r_{q^{\perp}}$, and $r_q^{\perp}$ commute with r.*

(viii) *$r_p = p \wedge r = pr = rp$, $r_{p^{\perp}} = p^{\perp} \wedge r = p^{\perp} r = rp^{\perp}$, $r_q = q \wedge r = qr = rq$, and $r_{q^{\perp}} = q^{\perp} \wedge r = q^{\perp} r = rq^{\perp}$.*

Proof. By Definition 3.7 and Theorem 3.3 (v), we have (i). Part (ii) follows from the facts that $r_p \leq p$, $r_{p^{\perp}} \leq p^{\perp}$, and $r = r_p + r_{p^{\perp}}$. Similar arguments prove (iii), (iv), and (v). Parts (vi) and (vii) follow from (ii)–(v).

Since $r = (p \vee q) \wedge (p \vee q^{\perp}) \wedge (p^{\perp} \vee q) \wedge (p^{\perp} \vee q^{\perp})$, it follows that $p \wedge r = p \wedge (p^{\perp} \vee q) \wedge (p^{\perp} \vee q^{\perp}) = r_p$, and $p \wedge r = pr = rp$ because pCr. The remaining equalities in (viii) are proved similarly. $\square$

4 Sine and cosine effect elements

4.1 Example. Again we consider the rank 2 synaptic algebra in Example 2.10, this time having a look at the situation of present interest in which $p, q \in P$. Assuming that $p, q \neq 0, 1$, and $p \neq q, q^{\perp}$, we can choose an orthonormal basis for $\mathcal{H}$ such that p diagonalizes and p, q, $p^{\perp}$, and $q^{\perp}$ are represented by the matrices

$$p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; q = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix},$$

$$p^{\perp} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \; q^{\perp} = \begin{bmatrix} \sin^2\theta & -\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos^2\theta \end{bmatrix},$$

where $0 < \theta < \frac{\pi}{2}$ is the positive acute angle between the one-dimensional subspaces upon which p and q project (see [10, p. 384]).

In Example 4.1, we have

$$pqp + p^{\perp}q^{\perp}p^{\perp} = (\cos^2\theta)I \text{ and } pq^{\perp}p + p^{\perp}qp^{\perp} = (\sin^2\theta)I,$$

respectively, where I is the identity matrix. This suggests the following definition.

4.2 Definition. As $0 \leq q, q^{\perp}$, we have $0 \leq pqp, p^{\perp}q^{\perp}p^{\perp}, pq^{\perp}p, p^{\perp}qp^{\perp}$, so $0 \leq pqp + p^{\perp}q^{\perp}p^{\perp}$ and $0 \leq pq^{\perp}p + p^{\perp}qp^{\perp}$, whence we define

$$c := (pqp + p^{\perp}q^{\perp}p^{\perp})^{1/2} \text{ and } s := (pq^{\perp}p + p^{\perp}qp^{\perp})^{1/2}.$$

We refer to c as the *cosine effect* and to s as the *sine effect* for the projection q with respect to p.

Recall that an element $e \in A$ is called an *effect* iff $0 \leq e \leq 1$. In part (vi) of the next theorem, we show that c and s are, in fact, effects in A.

4.3 Theorem.

(i) $c^2 = pqp + p^{\perp}q^{\perp}p^{\perp} = 1 - (p-q)^2 = (p - q^{\perp})^2 = (p + q - 1)^2$.

(ii) $s^2 = pq^{\perp}p + p^{\perp}qp^{\perp} = (p-q)^2$.

(iii) $pc^2 = pqp = c^2p, \; qc^2 = qpq = c^2q, \; ps^2 = pq^{\perp}p = s^2p,$
$qs^2 = qp^{\perp}q = s^2q,$ *and* $s^2p^{\perp} = p^{\perp}qp^{\perp} = p^{\perp}s^2$.

(iv) $c = |p - q^{\perp}|$ *and* $s = |p - q|$.

(v) $c^2 + s^2 = 1$.

(vi) $0 \leq c^2,\ s^2,\ c,\ s \leq 1$. *Also,* $c^2 \leq c$, *and* $s^2 \leq s$.

(vii) $C(c) = C(c^2) = C(s^2) = C(s)$.

(viii) cCp, cCq, cCr, sCp, sCq, sCr, *and* cCs.

(ix) $C(p) \cap C(q) \subseteq C(c) = C(s)$.

Proof. By direct calculation using Definition 4.2 and the facts that $p^{\perp} = 1-p$ and $q^{\perp} = 1-q$, we have $c^2 = pqp+p^{\perp}q^{\perp}p^{\perp} = 1-p+pq+qp-q = 1-(p-q)^2$. Also, $(p-q^{\perp})^2 = (p+q-1)^2 = 1-p+pq+qp-q$, and (i) follows. Similarly, $s^2 = pq^{\perp}p + p^{\perp}qp^{\perp} = p - pq - qp + q = (p-q)^2$, proving (ii).

Part (iii) follows by direct calculation using the facts that $c^2 = pqp + p^{\perp}q^{\perp}p^{\perp} = 1 - p + pq + qp - q$ and $s^2 = pq^{\perp}p + p^{\perp}qp^{\perp} = p - pq - qp + q$.

Part (iv) follows from (i), (ii), and the facts that $0 \leq c, s$.

Part (v) follows from $c^2 = 1-(q-p)^2 = 1-s^2$. Obviously, $0 \leq c^2,\ s^2,\ c,\ s$. Also, $c^2, s^2 \leq c^2 + s^2 = 1$. By [3, Corollary 3.4], the facts that $0 \leq s$, $0 \leq 1$, $sC1$, and $s^2 \leq 1 = 1^2$ imply that $s \leq 1$. Likewise, $c \leq 1$, so $0 \leq c,\ s,\ c^2,\ s^2 \leq 1$. That $c^2 \leq c$ and $s^2 \leq s$ then follow from [3, Lemma 2.5 (i)], and (vi) is proved.

Since $0 \leq c, s$ and $c^2 = 1-s^2$, it follows that $C(c) = C((c^2)^{1/2}) = C(c^2) = C(1-s^2) = C(s^2) = C((s^2)^{1/2}) = C(s)$, proving (vii).

By (iii) and (vii), $p, q \in C(c) = C(s)$, whence cCr and sCr. Also, $c \in C(c) = C(s)$, completing the proof of (viii).

By parts (iv) and (vii) above, $a \in C(p) \cap C(q) \Rightarrow a \in C(p - q^{\perp}) \Rightarrow a \in C(|p - q^{\perp}|) \Rightarrow a \in C(c) = C(s)$, proving (ix). $\square$

The following theorem concerns the carriers c^{o} and s^{o} of the cosine and sine effects c and s.

4.4 Theorem.

(i) $c^{\mathrm{o}}Cp$, $c^{\mathrm{o}}Cq$, $c^{\mathrm{o}}Cs$, $c^{\mathrm{o}}Cr$, $s^{\mathrm{o}}Cp$, $s^{\mathrm{o}}Cq$, $s^{\mathrm{o}}Cr$, $s^{\mathrm{o}}Cc$, *and* $c^{\mathrm{o}}Cs^{\mathrm{o}}$.

(ii) $c^{\mathrm{o}} = (p \vee q^{\perp}) \wedge (p^{\perp} \vee q)$ *and* $s^{\mathrm{o}} = (p \vee q) \wedge (p^{\perp} \vee q^{\perp})$.

(iii) $(cs)^{\mathrm{o}} = c^{\mathrm{o}}s^{\mathrm{o}} = c^{\mathrm{o}} \wedge s^{\mathrm{o}} = (p \vee q) \wedge (p \vee q^{\perp}) \wedge (p^{\perp} \vee q) \wedge (p^{\perp} \vee q^{\perp}) = r = [p, q]$.

(iv) $c^2 s^{o\perp} = s^{o\perp} c^2 = s^{o\perp}$, *whence* $s^{o\perp} \le c^2 \le c$.

(v) $s^2 c^{o\perp} = c^{o\perp} s^2 = c^{o\perp}$, *whence* $c^{o\perp} \le s^2 \le s$.

Proof. (i) Part (i) follows from Theorem 4.3 (viii), and the facts that $c^o \in CC(c)$ and $s^o \in CC(s)$.

(ii) Since $0 \le pqp,\ p^\perp q^\perp p^\perp$, we infer from Theorem 2.5 and Lemma 2.6 that

$$\begin{aligned} c^o &= (c^2)^o = (pqp + p^\perp q^\perp p^\perp)^o = (pqp)^o \vee (p^\perp q^\perp p^\perp)^o \\ &= (p \wedge (p^\perp \vee q)) \vee (p^\perp \wedge (p \vee q^\perp)) = (p \wedge (p^\perp \vee q)) \vee w, \end{aligned} \tag{1}$$

where $w := p^\perp \wedge (p \vee q^\perp)$. Now $pC(p^\perp \vee q)$ and pCw, whence

$$(p \wedge (p^\perp \vee q)) \vee w = (p \vee w) \wedge (p^\perp \vee q \vee w). \tag{2}$$

But $pCp^\perp$ and $pC(p \vee q^\perp)$, whence

$$p \vee w = p \vee (p^\perp \wedge (p \vee q^\perp)) = (p \vee p^\perp) \wedge (p \vee p \vee q^\perp) = p \vee q^\perp. \tag{3}$$

Furthermore, since $w \le p^\perp$,

$$p^\perp \vee q \vee w = p^\perp \vee q. \tag{4}$$

By Equations (3) and (4),

$$(p \vee w) \wedge (p^\perp \vee q \vee w) = (p \vee q^\perp) \wedge (p^\perp \vee q),$$

whence by Equations (2) and (1), $c^o = (p \vee q^\perp) \wedge (p^\perp \vee q)$.

Similarly,

$$s^o = (s^2)^o = ((pq^\perp p + p^\perp q p^\perp))^o,$$

and replacing q by $q^\perp$ in the calculations above, we find that $s^o = (p \vee q) \wedge (p^\perp \vee q^\perp)$, completing the proof of (ii).

(iii) Part (iii) follows from Theorem (ii), 4.3 (viii), Lemma 2.7, and Definitions 2.3 and 3.7.

(iv) Since $s^{o\perp} c^2 = s^{o\perp}(1 - s^2) = s^{o\perp} - 0 = s^{o\perp}$, we have $s^{o\perp} \le c^2 \le c$.

(iv) Since $c^{o\perp} s^2 = c^{o\perp}(1 - c^2) = c^{o\perp} - 0 = c^{o\perp}$, whence $c^{o\perp} \le s^2 \le s$. $\square$

Using Theorem 4.4, we obtain formulas for Halmos' four "thoroughly uninteresting" projections in terms of p, q, c, and s as follows.

4.5 Corollary.

(i) $s^{\mathrm{o}\perp}p = ps^{\mathrm{o}\perp} = s^{\mathrm{o}\perp}q = qs^{\mathrm{o}\perp} = s^{\mathrm{o}\perp} \wedge p = s^{\mathrm{o}\perp} \wedge q = p \wedge q$.

(ii) $c^{\mathrm{o}\perp}p = pc^{\mathrm{o}\perp} = c^{\mathrm{o}\perp}q^{\perp} = q^{\perp}c^{\mathrm{o}\perp} = c^{\mathrm{o}\perp} \wedge p = c^{\mathrm{o}\perp} \wedge q^{\perp} = p \wedge q^{\perp}$.

(iii) $c^{\mathrm{o}\perp}p^{\perp} = p^{\perp}c^{\mathrm{o}\perp} = c^{\mathrm{o}\perp}q = qc^{\mathrm{o}\perp} = c^{\mathrm{o}\perp} \wedge p^{\perp} = c^{\mathrm{o}\perp} \wedge q = p^{\perp} \wedge q$.

(iv) $s^{\mathrm{o}\perp}p^{\perp} = p^{\perp}s^{\mathrm{o}\perp} = s^{\mathrm{o}\perp}q^{\perp} = q^{\perp}s^{\mathrm{o}\perp} = s^{\mathrm{o}\perp} \wedge p^{\perp} = s^{\mathrm{o}\perp} \wedge q^{\perp} = p^{\perp} \wedge q^{\perp}$.

Proof. As a consequence of Theorem 4.4 (i), the projections $c^{\mathrm{o}\perp}$ and $s^{\mathrm{o}\perp}$ commute with both p and q. Also, by Theorem 4.4 (ii) and De Morgan,

$$c^{\mathrm{o}\perp} = (p^{\perp} \wedge q) \vee (p \wedge q^{\perp}) \text{ and } s^{\mathrm{o}\perp} = (p^{\perp} \wedge q^{\perp}) \vee (p \wedge q).$$

We prove (i). Proofs of (ii), (iii), and (iv) are similar. We have $s^{\mathrm{o}\perp}p = ps^{\mathrm{o}\perp} = p \wedge [(p^{\perp} \wedge q^{\perp}) \vee (p \wedge q)] = (p \wedge p^{\perp} \wedge q^{\perp}) \vee (p \wedge p \wedge q) = p \wedge q$. Likewise, $s^{\mathrm{o}\perp}q = qs^{\mathrm{o}\perp} = q \wedge [(p^{\perp} \wedge q^{\perp}) \vee (p \wedge q)] = (q \wedge p^{\perp} \wedge q^{\perp}) \vee (q \wedge p \wedge q) = p \wedge q$. $\square$

5 A general CS-decomposition theorem

We devote this section to a proof of a general CS-decomposition theorem that does not require the projections p and q to be in generic position (Theorem 5.6 below).

By Theorem 4.3 (iii), we have the following result.

5.1 Lemma. *The Peirce decomposition of q with respect to p takes the form*

$$q = pqp + pqp^{\perp} + p^{\perp}qp + p^{\perp}qp^{\perp} = c^2p + pqp^{\perp} + p^{\perp}qp + s^2p^{\perp}.$$

Thus, for the diagonal part of the Peirce decomposition of q with respect to p, we have

$$pqp + p^{\perp}qp^{\perp} = c^2p + s^2p^{\perp},$$

which is perfectly consistent with Halmos' Theorem 2 in [10], often called Halmos' *two projections theorem* or Halmos' *CS-decomposition theorem*. However, for full compliance with Halmos' theorem, we have to find a suitable formula, *in terms of the product cs*, for the off-diagonal part $pqp^{\perp} + p^{\perp}qp$ of the decomposition. (Note that Halmos' theorem was proved under the additional hypothesis that the projections involved are in generic position—see Section 6 below.) In this connection, the next theorem has a role to play.

5.2 Theorem.

(i) $c^2s^2 = pqp + qpq - pqpq - qpqp = p(qp^\perp q)p + p^\perp(qpq)p^\perp$
$= (pqp^\perp + p^\perp qp)^2$.

(ii) $cs = |pqp^\perp + p^\perp qp|$.

Proof.

(i) We have $c^2 = 1-(p-q)^2 = 1-p+pq+qp-q$ and $s^2 = (p-q)^2 = p-pq-qp+q$, and it follows by direct calculation that $c^2s^2 = pqp + qpq - pqpq - qpqp$. Also, direct calculations using the fact that $p^\perp = 1-p$ yield $pqp + qpq - pqpq - qpqp = p(qp^\perp q)p + p^\perp(qpq)p^\perp = (pqp^\perp + p^\perp qp)^2$.

(ii) As $cs = sc$, we have $(cs)^2 = c^2s^2$. Also, $0 \le c, s$ and $cs = sc$, so $0 \le cs$ by [3, Lemma 1.5], and (ii) then follows. □

5.3 Definition. As per Theorem 4.3 (iv) and Theorem 5.2 (ii), we define symmetries u, v, and k in A by polar decomposition of $p - q^\perp$, $p - q$, and $pqp^\perp + p^\perp qp$, respectively, as follows:

(1) $p - q^\perp = p + q - 1 = cu = uc$, where $u \in CC(p - q^\perp)$.

(2) $p - q = sv = vs$, where $v \in CC(p - q)$.

(3) $pqp^\perp + p^\perp qp = csk = kcs$, where $k \in CC(pqp^\perp + p^\perp qp) = p - ps - sp + s$.

5.4 Lemma. *The symmetries u, v, and k commute with both s and c.*

Proof. We already know that uCc; hence, since $C(c) = C(s)$ (Theorem 4.3 (vii)), we have uCs. Similarly, we already know that vCs, and therefore vCc.

We have cCp, cCq, sCp, sCq, so $c, s \in C(pqp^\perp + p^\perp qp)$. But $k \in CC(pqp^\perp + p^\perp qp)$, so kCc and kCs. □

5.5 Lemma.

(i) $upqpu = qpq$ *and* $u(p \wedge (p^\perp \vee q))u = q \wedge (q^\perp \vee p)$.

(ii) $vpq^\perp pv = q^\perp pq^\perp$ *and* $v(p \wedge (p^\perp \vee q^\perp)v = q^\perp \wedge (q \vee p)$.

(iii) $cs(pk + kp - k) = 0$ *and* $r(pk + kp - k) = 0$.

Proof. Part (i) follows from Theorem 2.8 and part (ii) follows from the same theorem upon replacing q by $q^\perp$.

To prove (iii), we begin by noting that since pCc, pCs, and $csk = pqp^\perp + p^\perp qp$, we have

$$cspk = pcsk = p(pep^\perp + p^\perp ep) = pep^\perp.$$

Moreover,

$$cskp^{\perp} = (pep^{\perp} + p^{\perp}ep)p^{\perp} = pep^{\perp},$$

and therefore

$$cs(pk + kp - k) = cs(pk - kp^{\perp}) = 0,$$

whence $r(pk + kp - k) = (cs)^{o}(pk + kp - k) = 0$ by Theorem 4.4 (iii). □

Combining Lemma 5.1, Definition 5.3, and Lemma 5.4, we obtain the following generalized version of Halmos' CS-decomposition theorem.

5.6 Theorem (Generalized CS-decomposition)**.**

$$q = c^2p + csk + s^2p^{\perp},$$

where $pqp = c^2p = pc^2$, $p^{\perp}qp^{\perp} = s^2p^{\perp} = p^{\perp}s^2$, $pqp^{\perp} + p^{\perp}qp = csk$, *$k$ is a symmetry,* cCs, cCk, sCk, *and* $k \in CC(pqp^{\perp} + p^{\perp}qp)$.

Note that we do not have to assume that p and q are in generic position in Theorem 5.6. However, at this point in the development of our theory, we do not have much information about the critical symmetry k involved in the formula $pqp^{\perp} + p^{\perp}qp = csk$ for the off-diagonal part of the Peirce decomposition of q with respect to p. Nevertheless, due to its generality, Theorem 5.6 can be useful.

5.7 Corollary. *Let* $p, q \in P$ *and let c and s be the cosine and sine effects for q with respect to p. Then the following conditions are mutually equivalent*:

(i) pCq.

(ii) *The off-diagonal part of q with respect to p is zero, i.e.,* $pqp^{\perp}+p^{\perp}qp = 0$.

(iii) $q = c^2p + s^2p^{\perp}$.

(iv) $cs = 0$

(v) *c and s are projections and* $c^{\perp} = s$.

(vi) $c, s \in P$, $c^{\perp} = s$, *and* $q = cp + c^{\perp}p^{\perp} = s^{\perp}p + sp^{\perp} = |p - s|$.

(vii) *There exists a projection* $t \in P$ *such that* tCp *and* $q = |p - t|$.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is Lemma 2.12 (ii). By Theorem 5.6, $q = c^2p + csk + s^2p^\perp$ where $csk = pqp^\perp + p^\perp qp$ and $k^2 = 1$, from which (ii) $\Leftrightarrow$ (iii) and (iii) $\Leftrightarrow$ (iv) both follow.

Now we claim that (iv) $\Rightarrow$ (v). Indeed, assume (iv). Then, since cCs, $0 = c^2s^2 = c^2(1-c^2) = c^2 - (c^2)^2$, so $c^2 \in P$, whence c is a partial symmetry with $0 \le c$. Thus, c is a projection, and by a similar argument, so is s; moreover, $c = c^2 = 1 - s^2 = 1 - s$, so $c^\perp = s$, and we have (iv) $\Rightarrow$ (v). Conversely, if (v) holds, then $cs = cc^\perp = 0$, so (iv) holds, and we have (iv) $\Leftrightarrow$ (v). Thus we have the mutual equivalence of conditions (i) through (v).

Assume (i). Then (iii), hence also (v) holds, whence $c, s \in P$, $c = s^\perp$, and $q = c^2p + s^2p^\perp = cp + c^\perp p^\perp = s^\perp p + sp^\perp = p - sp - ps + s = (p-s)^2$. Therefore, $q = q^{1/2} = |p - s|$. This proves that (i) $\Rightarrow$ (vi).

Obviously, with $t = s$, (vi) $\Rightarrow$ (vii), and it is clear that (vii) $\Rightarrow$ (i). □

A generalized CS-decomposition for the projection $q^\perp$ with respect to p is easily obtained from Theorem 5.6.

5.8 Corollary. $q^\perp = s^2p + cs(-k) + c^2p^\perp$.

Proof. We have $q^\perp = 1 - q = p + p^\perp - c^2p - csk - s^2p^\perp = (1-c^2)p + cs(-k) + (1-s^2)p^\perp = s^2p + cs(-k) + c^2p^\perp$. □

6 Generic position

As an immediate consequence of Definitions 2.1 and 2.3, we have the following.

6.1 Lemma. *p and q are in generic position iff $r = [p, q] = 1$.*

According to Theorem 3.8 (ii)–(v), the projections $r_p, r_{p^\perp}, r_q$ and $r_{q^\perp}$ belong to the lattice of projections $P[0, r]$ of the commutator algebra rAr of p and q. In this section we are going to prove that r_p and r_q are in generic position in rAr. We begin with two preliminary lemmas, the first of which—an immediate consequence of Theorem 3.8—identifies $r_{p^\perp}$ and $r_{q^\perp}$ as the orthocomplements of r_p and r_q in $P[0, r]$.

6.2 Lemma. (i) $r_p^{\perp_r} = r_p^\perp \wedge r = r_p^\perp r = rr_p^\perp = r_{p^\perp}$. (ii) $r_q^{\perp_r} = r_q^\perp \wedge r = r_q^\perp r = rr_q^\perp = r_{q^\perp}$.

6.3 Lemma. $r_p \wedge r_q = r_p \wedge r_{q^\perp} = r_{p^\perp} \wedge r_q = r_{p^\perp} \wedge r_{q^\perp} = 0$.

Proof. We have $r_p = p \wedge (p^\perp \vee q) \wedge (p^\perp \vee q^\perp) \leq p \wedge (p^\perp \vee q^\perp)$ and $r_q = q \wedge (p \vee q^\perp) \wedge (p^\perp \vee q^\perp) \leq q$, whence

$$r_p \wedge r_q \leq p \wedge (p^\perp \vee q^\perp) \wedge q = (p \wedge q) \wedge (p \wedge q)^\perp = 0,$$

whence $r_p \wedge r_q = 0$. The remaining equalities follow by symmetry. $\square$

6.4 Theorem. *The projections $r_p = pr = rp = r \wedge p \in P[0, r]$ and $r_q = qr = rq = r \wedge q \in P[0, r]$ are in generic position in the commutator algebra rAr.*

Proof. Combine Lemmas 6.2 and 6.3. $\square$

6.5 Corollary. $r_p \vee r_q = r_p \vee r_{q^\perp} = r_{p^\perp} \vee r_q = r_{p^\perp} \vee r_{q^\perp} = [p, q] = r.$

In view of Theorem 6.4, it seems natural to inquire about the cosine and sine effects of r_q with respect to r_p as calculated in rAr.

6.6 Definition.

$$c_r := (r_p r_q r_p + r_{p^\perp} r_{q^\perp} r_{p^\perp})^{1/2} \text{ and } s_r := (r_p r_{q^\perp} r_p + r_{p^\perp} r_q r_{p^\perp})^{1/2}.$$

6.7 Theorem.

(i) $c_r = cr = rc$ *and* $s_r = sr = rs$.

(ii) $c = c_r + |(p \wedge q) - (p^\perp \wedge q^\perp)|$ *and* $s = s_r + |(p \wedge q^\perp) - (p^\perp \wedge q)|$.

Proof. (i) By Theorem 4.3, we have $c_r = |r_p - r_{q^\perp}|$ and $s_r = |r_p - r_q|$. Thus, $c_r = |pr - q^\perp r| = |(p - q^\perp)r|$ and as $(p - q^\perp)Cr$ and $0 \leq r$, it follows that $c_r = |p - q^\perp||r| = |p - q^\perp|r = cr = rc$. Similarly, $s_r = |pr - qr| = |(p - q)r| = |p - q|r = sr = rs$.

(ii) As $r^\perp = (p \wedge q) \vee (p \wedge q^\perp) \vee (p^\perp \wedge q) \vee (p^\perp \wedge q^\perp) = (p \wedge q) + (p \wedge q^\perp) + (p^\perp \wedge q) + (p^\perp \wedge q^\perp)$, it follows that $pr^\perp = (p \wedge q) + (p \wedge q^\perp)$ and $q^\perp r^\perp = (p \wedge q^\perp) + (p^\perp \wedge q^\perp)$. Thus, $cr^\perp = |p - q^\perp|r^\perp = |pr^\perp - q^\perp r^\perp| = |(p \wedge q) + (p \wedge q^\perp) - (p \wedge q^\perp) - (p^\perp \wedge q^\perp)| = |(p \wedge q) - (p^\perp \wedge q^\perp)|$. Therefore, $c = cr + cr^\perp = c_r + |(p \wedge q) - (p^\perp \wedge q^\perp)|$. A similar calculation yields $s = s_r + |(p \wedge q^\perp) - (p^\perp \wedge q)|$. $\square$

7 Dropping down to the commutator algebra

If pCq, then $r = 0$ and by Theorems 3.3 and 3.4, $p, q, p^\perp$, and $q^\perp$ can be expressed in terms of Halmos' four "thoroughly uninteresting" projections, essentially concluding our study of p and q.

Using parts (i) and (iii) of Theorem 3.4, part (ii) of Theorem 6.7, and Halmos' four uninteresting projections, we can translate properties of r_p, r_q, c_r, and s_r into properties of p, q, c, and s; hence these theorems reduce the study of the two projections p and q in the synaptic algebra A to the study of the two projections r_p and r_q, which by Theorem 6.4 are in generic position in the commutator algebra rAr of p and q. As we are going to assume that p does not commute with q, i.e., $r \neq 0$, Corollary 3.5 will imply that r_p does not commute with r_q. Thus, we propose to drop down from A to the nondegenerate commutator algebra rAr and focus on the study of r_p and r_q in rAr. Consequently, to simplify notation, we shall now replace the synaptic algebra rAr by A and replace r_p and r_q by p and q, respectively. Notice that this is exactly what was done by Böttcher and Spitkovsky [2, p. 1414].

7.1 Standing Assumption. *In what follows, we assume that the two projections p and q are in generic position in the nondegenerate synaptic algebra A, i.e.,*

$$p \wedge q = p \wedge q^\perp = p^\perp \wedge q = p^\perp \wedge q^\perp = 0, \text{ and}$$
$$p \vee q = p \vee q^\perp = p^\perp \vee q = p^\perp \vee q^\perp = 1 \neq 0.$$

As a consequence of Assumption 7.1, $p = r_p$, $p^\perp = r_{p^\perp}$, $q = r_q$, $q^\perp = r_{q^\perp}$, and $r = [p, q] = 1$, so *we shall have no further use for r_p, $r_{p^\perp}$, r_q, $r_{q^\perp}$, r, and $[p, q]$.*

Notice that p and q are complements—but not orthocomplements—in the OML P. Likewise for p and $q^\perp$, $p^\perp$ and q, and $p^\perp$ and $q^\perp$.

7.2 Lemma. $(pqp)^{\mathrm{o}} = (pq^\perp p)^{\mathrm{o}} = p$, $(p^\perp qp^\perp)^{\mathrm{o}} = (p^\perp q^\perp p^\perp)^{\mathrm{o}} = p^\perp$, $(qpq)^{\mathrm{o}} = (qp^\perp q)^{\mathrm{o}} = q$, *and* $(q^\perp pq^\perp)^{\mathrm{o}} = (q^\perp p^\perp q^\perp)^{\mathrm{o}} = q^\perp$.

Proof. By Lemma 2.5 (ii), $(pqp)^{\mathrm{o}} = p \wedge (p^\perp \vee q) = p \wedge 1 = p$, and the remaining formulas follow similarly. □

7.3 Theorem. *The symmetries u and v* (Definition 5.3) *satisfy the following conditions*:

(i) *u exchanges p and q as well as $p^\perp$ and $q^\perp$.*

(ii) *v exchanges p and $q^\perp$ as well as q and $p^\perp$.*

Proof. In Lemma 5.5 we have $p \vee q = p \vee q^\perp = p^\perp \vee q = p^\perp \vee q^\perp = 1$ and it follows that $upu = q$ and $vpv = q^\perp$, whence $up^\perp u = q^\perp$ and $vqv = p^\perp$. $\square$

7.4 Definition. $j := uvp + pvu$ and $\ell := 2p - 1$.

7.5 Theorem.

(i) *j is a symmetry in A exchanging p and $p^\perp$.*

(ii) *j commutes with both s and c.*

(iii) *$j = pj + jp$.*

(iv) *$\ell = 2p - 1 = p - p^\perp = cu + sv$ is a symmetry that commutes with p, c, and s.*

Proof. (i) Put $x := uvp \in R$ and $y := pvu \in R$. Then $j = x + y \in A$. As $upu = q$, it follows that $up = upu^2 = qu$. Likewise, $uq^\perp = p^\perp u$, and therefore $x = uvp = uq^\perp v = p^\perp uv$. Similarly, $y = pvu = vq^\perp u = vup^\perp$. Consequently, $x^2 = y^2 = 0$, $xy = p^\perp uvvup^\perp = p^\perp$, and $yx = pvuuvp = p$; hence $j^2 = (x+y)^2 = x^2+xy+yx+y^2 = p^\perp + p = 1$, so j is a symmetry in A. Moreover, $xp = x$ and $yp = 0$, so $jpj = (x+y)p(x+y) = (xp+yp)(x+y) = x(x+y) = x^2 + xy = p^\perp$.

(ii) By Definition 5.3 (2), s commutes with v, by Lemma 5.4, s commutes with u, by Theorem 4.3 (viii), s commutes with p, and it follows that s commutes with $j = uvp + pvu$. A similar argument shows that c commutes with j.

(iii) As $1 = p^\perp + p = jpj + p$, it follows that $j = j^2pj + jp = pj + jp$.

(iv) Evidently, $cu + sv = (p - q^\perp) + (p - q) = 2p - 1 = \ell$ and $(2p-1)^2 = 4p - 4p + 1 = 1$, so ℓ is a symmetry. Obviously, $\ell \in C(p) \cap C(c) \cap C(s)$. $\square$

7.6 Example. In the rank 2 synaptic algebra in Example 4.1, the projections p and q are in generic position. For the symmetries u, v, j, and ℓ, we have

$$u = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}, \quad v = \begin{bmatrix} \sin\theta & -\cos\theta \\ -\cos\theta & -\sin\theta \end{bmatrix},$$

$$j = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } \ell = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

7.7 Lemma.

(i) $c^{\mathrm{o}} = s^{\mathrm{o}} = (cs)^{\mathrm{o}} = (csj)^{\mathrm{o}} = 1$.

(ii) $uv + vu = 0$.

(iii) $j = k$.

Proof. (i) By Theorem 4.4 (ii), $c^{\mathrm{o}} = (p \vee q^{\perp}) \wedge (p^{\perp} \vee q) = 1 \wedge 1 = 1$ and $s^{\mathrm{o}} = (p \vee q) \wedge (p^{\perp} \vee q^{\perp}) = 1 \wedge 1 = 1$. Also, $cs = sc \in A$, so by [8, Theorem 5.5], $(cs)^{\mathrm{o}} = c^{\mathrm{o}}s^{\mathrm{o}} = 1$. Moreover, $j^{\mathrm{o}} = (j^2)^{\mathrm{o}} = 1^{\mathrm{o}} = 1$, and $csj = jcs$, so by [8, Theorem 5.5] again, $(csj)^{\mathrm{o}} = (cs)^{\mathrm{o}}j^{\mathrm{o}} = 1$.

(ii) By Theorem 7.5 (iv), $1 = \ell^2 = (cu+sv)^2 = (cu)^2+cs(uv+vu)+(sv)^2 = c^2 + s^2 + cs(uv + vu) = 1 + cs(uv + vu)$, whence $cs(uv + vu) = 0$, and it follows from (i) that $0 = (cs)^{\mathrm{o}}(uv + vu) = 1(uv + vu) = uv + vu$.

(iii) By Lemma 4.8 and Theorem 4.3 (viii), c and s commute with u, v, p and each other, whence by Definition 5.3, and direct calculation

$$csj = cs(uvp + pvu) = csuvp + cspvu = (cu)(sv)p + p(sv)(cu)$$

$$= (p + q - 1)(p - q)p + p(p - q)(p + q - 1) = pq + qp - 2pqp = pqp^{\perp} + p^{\perp}qp.$$

Also, by Theorem 5.6, we have $csk = pqp^{\perp} + p^{\perp}qp$, and it follows that $cs(k - j) = 0$. Thus, by (i), $k - j = (cs)^{\mathrm{o}}(k - j) = 0$, proving (iii). $\square$

Combining Theorem 5.6, Theorem 7.5, and Lemma 7.7, we obtain the following synaptic-algebra version of Halmos' CS-decomposition theorem.

7.8 Theorem (CS-Decomposition)**.** *If p and q are projections in generic position in A, then*

$$q = c^2p + csj + s^2p^{\perp},$$

where $pqp = c^2p = pc^2$, $p^{\perp}qp^{\perp} = s^2p^{\perp} = p^{\perp}s^2$, $pqp^{\perp} + p^{\perp}qp = csj$, $c^{\mathrm{o}} = s^{\mathrm{o}} = 1$, j is a symmetry exchanging p and $p^{\perp}$, cCs, cCj, sCj, and $j \in CC(pqp^{\perp} + p^{\perp}qp)$.

In Section 9 we show that Halmos' CS-decomposition theorem can be derived from Theorem 7.8.

8 Applications of the CS-decomposition

In this section we illustrate the utility of Theorem 7.8 by establishing some results analogous to those in [2] and [10]. Thus, in what follows, *we assume that p and q are projections in generic position and that the CS-decomposition of q with respect to p is*

$$q = c^2p + csj + s^2p^\perp.$$

In the following theorem we use Theorem 7.8 to calculate the spectrum of the sum $p+q$. We denote by $\sigma(a)$ the spectrum of an element $a \in A$ and, as is customary, we identify each real number $\lambda \in \mathbb{R}$ with the element $\lambda 1 \in A$. See [3, §8] for an account of spectral theory in a synaptic algebra.

8.1 Theorem (Cf. [2, Example 2.1])**.** *The spectrum of $p+q$ is $\sigma(p+q) = \{1 \pm \gamma : \gamma \in \sigma(c)\}$.*

Proof. Put $a := p+q$. Then by Theorem 4.3 (i), $(a-1)^2 = (p+q-1)^2 = c^2$, whereupon

$$\{(\lambda-1)^2 : \lambda \in \sigma(a)\} = \sigma((a-1)^2) = \sigma(c^2) = \{\gamma^2 : \gamma \in \sigma(c)\}.$$

Therefore, for all $\lambda \in \sigma(p+q) = \sigma(a)$, there exists $\gamma \in \sigma(c)$ such that $\lambda = 1+\gamma$ or $\lambda = 1-\gamma$. Moreover, for any $\gamma \in \sigma(c)$ there exists $\lambda \in \sigma(p+q)$ such that one of the latter two equations holds.

Let $\gamma \in \sigma(c)$. To complete the proof, it will suffice to show that $1+\gamma \in \sigma(p+q)$ iff $1-\gamma \in \sigma(p+q)$, i.e., that $\gamma \in \sigma(p+q-1)$ iff $-\gamma \in \sigma(p+q-1)$. By the CS-decomposition of q with respect to p, we have

$$\begin{aligned} p+q-1 &= p + c^2p + s^2p^\perp + csj - p - p^\perp = c^2p + (s^2-1)p^\perp + csj \\ &= c^2p - c^2p^\perp + csj = c^2(p-p^\perp) + csj = c^2\ell + csj, \end{aligned}$$

where $\ell := p - p^\perp = 2p-1$ is a symmetry commuting with p, c, and s (Theorem 7.5 (iv)); moreover, from $jpj = p^\perp$ we get

$$j\ell j = -\ell, \ \ \ell j = -j\ell, \ \text{ and } \ell j \ell = -j.$$

Thus, the element $(p+q-1)-\gamma = c^2\ell + csj - \gamma$ is invertible iff $j(c^2\ell + csj - \gamma)j = -c^2\ell + csj - \gamma$ is invertible iff $\ell(-c^2\ell + csj - \gamma)\ell = -c^2\ell - csj - \gamma = -(p+q-1)-\gamma$ is invertible. Hence, $\gamma \in \sigma(p+q-1)$ iff $-\gamma \in \sigma(p+q-1)$. □

We now turn our attention to some commutativity results that involve the CS-decomposition.

8.2 Lemma. *Suppose that there exists $b = bp = pb \in C(c)$ and that $a = b + jbj$. Then $ap = pa = b$ and $a \in C(q)$.*

Proof. Assume the hypotheses of the lemma. Then

$$bp^{\perp} = p^{\perp}b = 0, \text{ so } jbjp = jbp^{\perp}j = 0 = jp^{\perp}bj = pjbj, \tag{1}$$

whence

$$jbjp^{\perp} = p^{\perp}jbj = jbj, \text{ and } ap = bp + jbjp = b = pb + pjbj = pa. \tag{2}$$

Since $b \in C(c)$, we have $b \in C(s)$ by Theorem 4.3 (viii). Using the data in (1) and (2), we find that

$$aq = b(pc^2 + csj + p^{\perp}s^2) + jbj(pc^2 + jcs + p^{\perp}s^2) = c^2b + csbj + jbcs + jbjs^2,$$

whereas

$$qa = (c^2p + jcs + s^2p^{\perp})b + (c^2p + csj + s^2p^{\perp})jbj = c^2b + jbcs + csbj + s^2jbj.$$

Since s^2Cjbj, it follows that $aq = qa$. $\square$

8.3 Theorem. *Let $z \in P$ be a projection. Then $z \in C(p) \cap C(q)$ iff there exists a projection $t \in P$ such that $t = tp = pt \in C(c)$ and $z = t + jtj$.*

Proof. If $t \in P$, $t = tp = pt \in C(c)$, and $z = t + jtj$, then $z \in C(p) \cap C(q)$ by Lemma 8.2 with $a := z$ and $b := t$.

Conversely, suppose that $z \in P \cap C(p) \cap C(q)$ and let $g := |p - z^{\perp}|$ be the cosine effect of the projection z with respect to p (Theorem 4.3 (iv)). Thus, $g \in C(p)$ and since z commutes with p, we infer from Corollary 5.7 (vi) that g is a projection and $z = gp + g^{\perp}p^{\perp}$. By Theorem 4.3 (ix), $z \in C(c)$. Moreover, as $p, z \in C(c)$, we have $p - z^{\perp} \in C(c)$, whence $g = |p - z^{\perp}| \in C(c)$. Also, since $j \in CC(pqp^{\perp} + p^{\perp}qp)$ and $z \in C(p) \cap C(q)$, it follows that $j \in C(z)$. From this and from $p^{\perp}j = jp$ we find that

$$gp + g^{\perp}p^{\perp} = z = jzj = jgpj + jg^{\perp}p^{\perp}j = jgjp^{\perp} + jg^{\perp}jp,$$

and multiplying both sides of the last equation by $p^{\perp}$ from the right, we obtain $g^{\perp}p^{\perp} = jgjp^{\perp}$. Consequently,

$$z = gp + g^{\perp}p^{\perp} = gp + jgjp^{\perp} = gp + jgpj.$$

Now put $t := gp = pg$. Then $z = t + jtj$, $tp = pt = t$, and $t^2 = g^2p = gp = t$, so $t \in P$. Moreover, since $g, p \in C(c)$, it follows that $t \in C(c)$. $\square$

In the next theorem, we find conditions under which an arbitrary element $a \in A$ commutes with projections p and q in generic position. The limits in the proof are taken with respect to the order-unit norm on A [3, p. 634].

8.4 Theorem. *An element $a \in A$ commutes with both p and q iff there exists $b \in C(c)$ such that $b = bp = pb$ and $a = b + jbj$.*

Proof. If $b \in C(c)$, $b = bp = pb$, and $a = b + jbj$, then $a \in C(p) \cap C(q)$ by Lemma 8.2.

Conversely, assume that $a \in C(p) \cap C(q)$ and let $(z_\lambda)_{\lambda \in \mathbb{R}}$ be the spectral resolution of a ([3, Definition 8.2 (ii)]). By [3, Theorem 8.10], $z_\lambda \in P \cap C(p) \cap C(q)$ for all $\lambda \in \mathbb{R}$, whence by Theorem 8.3, for each $\lambda \in \mathbb{R}$, there exists a projection $t_\lambda \in P$ such that $t_\lambda = t_\lambda p = pt_\lambda \in C(c)$ and $z_\lambda = t_\lambda + jt_\lambda j$.

By [3, Corollary 8.6], there is an ascending sequence $a_1 \leq a_2 \leq \cdots$ in $CC(a)$ such that $a = \lim_{n\to\infty} a_n$ and each a_n is a finite real linear combination of projections z_λ in the spectral resolution of a. Let n be a positive integer. Then, since $a_n \in CC(a)$ and $a \in C(p) \cap C(q)$, it follows that $a_n \in C(p) \cap C(q)$. Moreover,

$$a_n = \sum_{i=1}^{M_n} \alpha_{n,i} z_{\lambda_{n,i}} = \sum_{i=1}^{M_n} \alpha_{n,i}(t_{\lambda_{n,i}} + jt_{\lambda_{n,i}} j) = d_n + jd_n j,$$

where $\alpha_{n,i} \in \mathbb{R}$ and $d_n := \sum_{i=1}^{M_n} \alpha_{n,i} t_{\lambda_{n,i}}$. Since $t_{\lambda_{n,i}} = t_{\lambda_{n,i}} p = pt_{\lambda_{n,i}} \in C(c)$, we have

$$d_n = d_n p = pd_n \in C(c), \text{ and } jd_n j = jd_n pj = jd_n jp^\perp.$$

Thus, $a_n p = pa_n = d_n p + jd_n jp = d_n \in C(c)$. Put $b := ap = pa$, noting that $b = bp = pb$. Also, since $a, p \in C(c)$, we have $b \in C(c)$. Moreover,

$$b = ap = (\lim_{n\to\infty} a_n)p = \lim_{n\to\infty}(a_n p) = \lim_{n\to\infty} d_n.$$

By [3, Theorem 8.11], $C(c)$ is closed in the order-unit-norm topology, whence $b = \lim_{n\to\infty} d_n \in C(c)$. Moreover, since j is a symmetry, we have

$$jbj = j(\lim_{n\to\infty} d_n)j = \lim_{n\to\infty}(jd_n j),$$

and it follows that

$$a = \lim_{n\to\infty} a_n = \lim_{n\to\infty}(d_n + jd_n j) = b + jbj. \qquad \square$$

9 Operator-matrix consequences

Let $\mathcal{H}$ be a nonzero complex separable Hilbert space, let $\mathcal{B}(\mathcal{H})$ be the C*-algebra of all bounded linear operators on $\mathcal{H}$, and let A be the synaptic algebra of all self-adjoint operators in $\mathcal{B}(\mathcal{H})$. The assumption that $p, q \in A$ are projections in generic position is still in force. Following the notation in [2, pp. 1413 and ff.], we define the following closed linear subspaces of $\mathcal{H}$:

$$M_0 := p(\mathcal{H}) \text{ and } M_1 := p^{\perp}(\mathcal{H}).$$

Then $\mathcal{H} = M_0 \oplus M_1$; hence, in what follows, *we shall regard each vector $h \in \mathcal{H}$ as having the form*

$$h = \begin{bmatrix} x \\ w \end{bmatrix}, \text{ where } x \in M_0,\ w \in M_1,\ ph = \begin{bmatrix} x \\ 0 \end{bmatrix} \text{ and } p^{\perp}h = \begin{bmatrix} 0 \\ w \end{bmatrix}.$$

By Theorem 7.5, p and $p^{\perp}$ are exchanged by a symmetry j in A. Thus, for each $w \in M_1$,

$$j\begin{bmatrix} 0 \\ w \end{bmatrix} = jp^{\perp}\begin{bmatrix} 0 \\ w \end{bmatrix} = pj\begin{bmatrix} 0 \\ w \end{bmatrix},$$

whence there is a uniquely determined element $Rw \in M_0$ such that

$$j\begin{bmatrix} 0 \\ w \end{bmatrix} = \begin{bmatrix} Rw \\ 0 \end{bmatrix}.$$

It is not difficult to show that $R^* = R^{-1}$, i.e., $R\colon M_1 \to M_0$ is a unitary isomorphism, and $R^*\colon M_0 \to M_1$ satisfies

$$j\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ R^*x \end{bmatrix}, \text{ whence } j\begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} Rw \\ R^*x \end{bmatrix} = \begin{bmatrix} 0 & R \\ R^* & 0 \end{bmatrix}\begin{bmatrix} x \\ w \end{bmatrix}.$$

Thus in operator-matrix form,

$$j = \begin{bmatrix} 0 & R \\ R^* & 0 \end{bmatrix} \text{ and } p = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

where $I\colon M_0 \to M_0$ is the identity operator.

Let

$$\mathcal{K} := \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x, y \in M_0 \right\}$$

be organized into a Hilbert space in the obvious way. It is easy to verify that the mappings given in operator-matrix form by

$$\begin{bmatrix} I & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} x \\ Rw \end{bmatrix} \text{ and } \begin{bmatrix} I & 0 \\ 0 & R^* \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ R^*y \end{bmatrix}$$

for

$$\begin{bmatrix} x \\ w \end{bmatrix} \in \mathcal{H} \text{ and } \begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{K}$$

are inverse unitary isomorphisms of $\mathcal{H}$ onto $\mathcal{K}$ and of $\mathcal{K}$ onto $\mathcal{H}$.

Since the projections p and q are in generic position, Theorem 7.8 yields the decomposition

$$q = c^2p + csj + s^2p^\perp,$$

where c and s are the sine and cosine effects of q with respect to p, $pqp = c^2p = pc^2$, $p^\perp qp^\perp = s^2p^\perp = p^\perp s^2$, cCp, sCp, cCs, and as above, j is a symmetry exchanging p and $p^\perp$. Moreover, $pqp^\perp + p^\perp qp = csj$, cCj, sCj, and $j \in CC(pqp^\perp + p^\perp qp)$.

For each $x \in M_0$,

$$c\begin{bmatrix} x \\ 0 \end{bmatrix} = cp\begin{bmatrix} x \\ 0 \end{bmatrix} = pc\begin{bmatrix} x \\ 0 \end{bmatrix} \text{ and } s\begin{bmatrix} x \\ 0 \end{bmatrix} = sp\begin{bmatrix} x \\ 0 \end{bmatrix} = ps\begin{bmatrix} x \\ 0 \end{bmatrix}$$

whence there are uniquely determined elements $Cx \in M_0$ and $Sx \in M_0$ such that

$$c\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} Cx \\ 0 \end{bmatrix} \text{ and } s\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} Sx \\ 0 \end{bmatrix}.$$

Using the fact that cCj, we have, for all $w \in M_1$,

$$c\begin{bmatrix} 0 \\ w \end{bmatrix} = cj\begin{bmatrix} Rw \\ 0 \end{bmatrix} = jc\begin{bmatrix} Rw \\ 0 \end{bmatrix} = j\begin{bmatrix} CRw \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ R^*CRw \end{bmatrix},$$

whence for $x \in M_0$ and $w \in M_1$,

$$c\begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} Cx \\ R^*CRw \end{bmatrix}, \text{ similarly } s\begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} Sx \\ R^*SRw \end{bmatrix},$$

and it follows that

$$csj\begin{bmatrix} x \\ w \end{bmatrix} = cs\begin{bmatrix} Rw \\ R^*x \end{bmatrix} = \begin{bmatrix} CSRw \\ R^*CSRR^*x \end{bmatrix} = \begin{bmatrix} CSRw \\ R^*CSx \end{bmatrix}.$$

Using the properties of c and s, it is not difficult to show that C and S are self-adjoint operators on M_0, $0 \le C \le I$, $0 \le S \le I$, $C^2 + S^2 = I$, and that C and S have kernel zero. We note that, in operator-matrix form,

$$c = \begin{bmatrix} C & 0 \\ 0 & R^*CR \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & R^* \end{bmatrix} \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & R \end{bmatrix} \text{ and}$$

$$s = \begin{bmatrix} S & 0 \\ 0 & R^*SR \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & R^* \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & R \end{bmatrix}.$$

Similarly, it is not difficult to show that the symmetries u and v can be expressed in matrix-operator form as

$$u = \begin{bmatrix} C & SR \\ R^*S & -R^*CR \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & R^* \end{bmatrix} \begin{bmatrix} C & S \\ S & -C \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & R \end{bmatrix} \text{ and}$$

$$v = \begin{bmatrix} S & -CR \\ -R^*C & -R^*SR \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & R^* \end{bmatrix} \begin{bmatrix} S & -C \\ -C & -S \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & R \end{bmatrix}.$$

In view of the results above,

$$q \begin{bmatrix} x \\ w \end{bmatrix} = (c^2p + csj + s^2p^\perp) \begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} C^2x + CSRw \\ R^*CSx + R^*SRw \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 \\ 0 & R^* \end{bmatrix} \begin{bmatrix} C^2 & CS \\ CS & S^2 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix},$$

whereupon, in operator-matrix form,

$$q = \begin{bmatrix} I & 0 \\ 0 & R^* \end{bmatrix} \begin{bmatrix} C^2 & CS \\ CS & S^2 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & R \end{bmatrix}.$$

This is precisely [2, Theorem 1.1]; hence, *Theorem 7.8 is a true generalization of Halmos' CS-decomposition theorem.*

Now let $a \in A$. Then by Theorem 8.4, $a \in C(p) \cap C(q)$ iff there exists $b \in C(c)$ such that $a = b + jbj$ and $b = bp = pb$. Evidently, $b = bp = pb$ iff b has the operator-matrix form

$$b = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix},$$

where B is a self-adjoint operator on M_0, in which case

$$jbj = \begin{bmatrix} 0 & R \\ R^* & 0 \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & R \\ R^* & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & R^*BR \end{bmatrix}$$

and bCc iff $BC = CB$. Moreover, in operator-matrix form, the condition $a = b + jbj$ is

$$a = \left[\begin{array}{cc} B & 0 \\ 0 & 0 \end{array}\right] + \left[\begin{array}{cc} 0 & 0 \\ 0 & R^*BR \end{array}\right] = \left[\begin{array}{cc} 0 & 0 \\ 0 & R^* \end{array}\right] \left[\begin{array}{cc} B & 0 \\ 0 & B \end{array}\right] \left[\begin{array}{cc} 0 & 0 \\ 0 & R \end{array}\right].$$

This is precisely Halmos' solution of the problem of finding the simultaneous commutant of two projections in generic position [10, p.385].

References

[1] Beran, L., *Orthomodular Lattices, An Algebraic Approach*, Mathematics and its Applications, Vol. 18, D. Reidel Publishing Company, Dordrecht, 1985.

[2] Böttcher, A. and Spitkovsky, I.M., A gentle guide to the basics of two projections theory, *Linear Algebra Appl.* **432**, no. 6 (2010) 1412–1459.

[3] Foulis, D.J., Synaptic algebras, *Math. Slovaca* **60**, no. 5 (2010) 631–654.

[4] Foulis, D.J. and Pulmannová, Spin factors as generalized Hermitian algebras, *Found. Phys.* **39**, no. 3 (2009) 237–255.

[5] Foulis, D.J. and Pulmannová, S., Projections in synaptic algebras, *Order* **27**, no. 2, (2010), 235–257.

[6] Foulis, D.J. and Pulmannová, S., Type-decomposition of a synaptic algebra, *Found. Phys.* **43**, no. 8, (2013), 948–968.

[7] Foulis, D.J. and Pulmannová, S., Symmetries in synaptic algebras. *Math. Slovaca* **64**, no. 3, (2014), 751–776.

[8] Foulis, D.J. and Pulmannová, S., Commutativity in a synaptic algebra. to appear in Math. Slovaca.

[9] Gudder, S., Pulmannová, S., Bugajski, S., and Beltrametti, E., Convex and linear effect algebras, *Rep. Math. Phys.* **44**, No. 3 (1999) 359–379.

[10] Halmos, P., Two subspaces, *Trans. Amer. Math. Soc.* **144** (1969) 381–389.

[11] Kalmbach, G., *Orthomodular Lattices*, Academic Press, Inc., London/New York, 1983.

[12] Marsden, E.L. Jr., The commutator and solvability in a generalized orthomodular lattice, *Pacific J. Math.* **33** (1870) 357–361.

[13] Pulmannová, S., A note on ideals in synaptic algebras, *Math. Slovaca* **62**, no. 6 (2012) 1091-?1104.

[14] Topping, D.M., *Jordan Algebras of Self-Adjoint Operators*, A.M.S. Memoir **No 53** AMS, Providence, Rhode Island, 1965.